九十七學年第一學期 PHYS2310 電磁學 期末考試題(共兩頁)

[Griffiths Ch. 4-6]　2009/01/15,　10:10am–12:00am, 教師：張存續

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Useful formulas: Cylindrical coordinate $\nabla\times\mathbf{v}=[\frac{1}{s}\frac{\partial v_z}{\partial\phi}-\frac{\partial v_\phi}{\partial z}]\hat{\mathbf{s}}+[\frac{\partial v_s}{\partial z}-\frac{\partial v_z}{\partial s}]\hat{\phi}+\frac{1}{s}[\frac{\partial(sv_\phi)}{\partial s}-\frac{\partial v_s}{\partial\phi}]\hat{\mathbf{z}}$

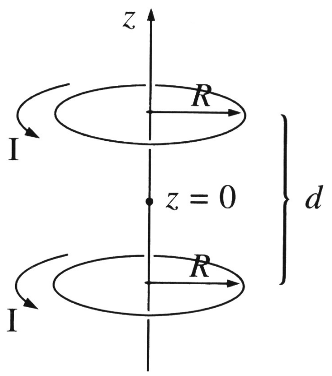
◇ Specify the magnitude and direction for a vector field.

1. (10%, 10%) The magnetic field on the axis of a circular current loop is far from uniform. We can produce a more nearly uniform field by using two such circular loops a distance *d* apart.

(a) Find the total magnetic field **B** along the *z*-axis as a function of *z*.

(b) Show that $\partial B/\partial z$ is zero at the point midway between them.

(c) Determine *d* such that $\partial B^2/\partial z^2=0$ at the midpoint, and find the resulting magnetic field at the center (*z*=0). [Hint: This arrangement is known as a Helmholtz coil.]



2. (4% x 5) The space between the planes of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness *a*, so the total distance between the plates is 2*a*. Slab 1 has a dielectric constant of 9, and slab 2 has a dielectric constant of 4. The free charge density on the top plate is $+\sigma$ and on the bottom plate is $-\sigma$.

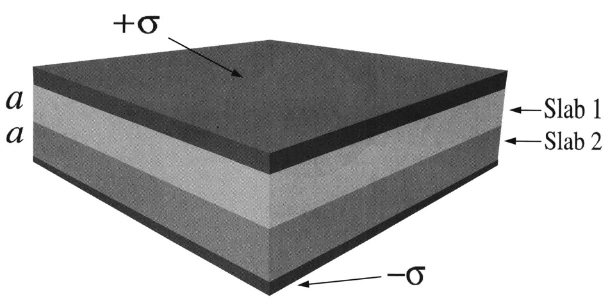
(a) Find the electric displacement **D** and the electric field **E** in each slab.

(b) Find the polarization **P** in each slab.

(c) Find the potential difference between the metal plates.

(d) Find the location and amount of all bound charges σ_b and ρ_b.

(e) Now that you know all the charges (free and bound), recalculate the field **D** and **E** in each slab, and confirm your answer to (a).



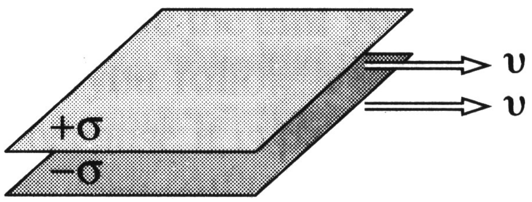
3. (10%,10%)

(a) Write down and prove the electrostatic boundary conditions in terms of **D**, **P**, and σ_f.

(b) Write down and prove the magnetostatic boundary condition in terms of **H**, **M**, and $\mathbf{K}_f$.

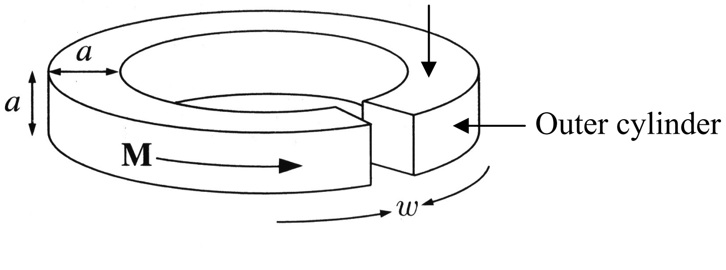
[Hint: write down the differential equations and their integral forms, and then use the divergence theorem and Stokes' theorem to prove them.]

4. (7%, 7%, 6%) A large parallel-plate capacitor, with uniform surface charge $+\sigma$ on the upper plate and $-\sigma$ on the lower, is moving with a constant speed v, as shown in the figure.
   (a) Find the magnetic field between the plates and also above and below them.
   (b) Find the magnetic force per unit area on the lower plate (attractive or repulsive force).
   (c) At what speed v would the magnetic force balance the electric force?



5. (10%,10%) An iron rod of length L and square cross section (side a), is given a uniform longitudinal magnetization **M**, and then bent around into a circle with a narrow gap (width w), as shown in the figure. ($\mathbf{M} = M_0\hat{\phi}$)
   (a) Find the $\mathbf{J}_b$ and $\mathbf{K}_b$ on the outer cylinder and the top surface.
   (b) Find the magnetic field at the center of the gap, assuming $w<a<<L$.

Top surface



1.

(a) $\mathbf{B}(r)=\dfrac{\mu_0 I}{4\pi}\int\dfrac{d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$. In the diagram only the z-component survives.

Upper coil: $B_{z,up}(z)=\dfrac{\mu_0 I}{4\pi}\dfrac{2\pi a\sin\theta}{\left((z-\frac{d}{2})^2+a^2\right)}\hat{\mathbf{z}}$, where $\sin\theta=\dfrac{a}{\sqrt{(z-\frac{d}{2})^2+a^2}}$

Total field: $B_z(z)=B_{z,up}+B_{z,down}=\dfrac{\mu_0 IR^2}{2}\left[\left((z-\frac{d}{2})^2+R^2\right)^{-\frac{3}{2}}+\left((z+\frac{d}{2})^2+R^2\right)^{-\frac{3}{2}}\right]$ in z-direction. #

(b) $\dfrac{\partial B_z(z)}{\partial z}=\dfrac{\mu_0 IR^2}{2}\left[-3(z-\frac{d}{2})\left((z-\frac{d}{2})^2+R^2\right)^{-\frac{5}{2}}-3(z+\frac{d}{2})\left((z+\frac{d}{2})^2+R^2\right)^{-\frac{5}{2}}\right]$.

$$\frac{\partial B_z(z=0)}{\partial z}=\frac{\mu_0 IR^2}{2}\left[3(\tfrac{d}{2})\left((\tfrac{d}{2})^2+R^2\right)^{-\frac{5}{2}}-3(\tfrac{d}{2})\left((\tfrac{d}{2})^2+R^2\right)^{-\frac{5}{2}}\right]=0 \quad \#$$

(c)

$$\frac{\partial^2 B_z(z)}{\partial z^2}=\frac{\mu_0 IR^2}{2}\left[-3\left((z-\tfrac{d}{2})^2+R^2\right)^{-\frac{5}{2}}-3\left((z+\tfrac{d}{2})^2+R^2\right)^{-\frac{5}{2}}+15(z-\tfrac{d}{2})^2\left((z-\tfrac{d}{2})^2+R^2\right)^{-\frac{7}{2}}+15(z+\tfrac{d}{2})^2\left((z+\tfrac{d}{2})^2+R^2\right)^{-\frac{7}{2}}\right]$$

$$=\frac{3\mu_0 IR^2}{2}\left[\frac{5(z-\frac{d}{2})^2-(z-\frac{d}{2})^2+R^2}{\left((z-\frac{d}{2})^2+R^2\right)^{\frac{7}{2}}}+\frac{5(z+\frac{d}{2})^2-(z+\frac{d}{2})^2+R^2}{\left((z+\frac{d}{2})^2+R^2\right)^{\frac{7}{2}}}\right]$$

$$\frac{\partial^2 B_z(z=0)}{\partial z^2}=\frac{3\mu_0 IR^2}{2}\left[\frac{-d^2+R^2}{\left((\frac{d}{2})^2+R^2\right)^{\frac{7}{2}}}+\frac{-d^2+R^2}{\left((\frac{d}{2})^2+R^2\right)^{\frac{7}{2}}}\right]=3\mu_0 IR^2\frac{-d^2+R^2}{\left((\frac{d}{2})^2+R^2\right)^{\frac{7}{2}}}$$

$\Rightarrow\ d=R$. The two coils are separated by a distance R. #

$$B_z(z)=\frac{\mu_0 IR^2}{2}\left[\left((\tfrac{R}{2})^2+R^2\right)^{-\frac{3}{2}}+\left((\tfrac{R}{2})^2+R^2\right)^{-\frac{3}{2}}\right]=\frac{8\mu_0 I}{5^{3/2}R}$$ in z-direction.

2.

(a) $\nabla\cdot\mathbf{D}=\rho_f\Rightarrow\oint\mathbf{D}\cdot d\mathbf{a}=Q_{f\,\text{enc}}$

Draw a cylindrical rectangular Gaussian surface, of area a, and apply the Gauss's law.

We find $Da=a\sigma$, $\mathbf{D}=-\sigma\hat{\mathbf{z}}$ in slab 1 and slab 2. $\mathbf{E}=\dfrac{1}{\varepsilon}\mathbf{D}=\dfrac{1}{\varepsilon_0\varepsilon_r}\mathbf{D}=\begin{cases}-\dfrac{\sigma}{9\varepsilon_0}\hat{\mathbf{z}} & \text{slab 1}\\ -\dfrac{\sigma}{4\varepsilon_0}\hat{\mathbf{z}} & \text{slab 2}\end{cases}$

(b) $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$, where $\mathbf{E} = \begin{cases} -\dfrac{\sigma}{9\varepsilon_0}\hat{\mathbf{z}} & \text{slab 1} \\ -\dfrac{\sigma}{4\varepsilon_0}\hat{\mathbf{z}} & \text{slab 2} \end{cases} \Rightarrow \mathbf{P} = \begin{cases} -\dfrac{8\sigma}{9}\hat{\mathbf{z}} & \text{slab 1} \\ -\dfrac{3\sigma}{4}\hat{\mathbf{z}} & \text{slab 2} \end{cases}$

(c) The potential difference between the metal plates

$$V = -\int_0^{2a} \mathbf{E}\cdot d\mathbf{l} = -\int_0^{a} \mathbf{E}_1\cdot d\mathbf{l} - \int_a^{2a} \mathbf{E}_2\cdot d\mathbf{l} = \frac{\sigma a}{9\varepsilon_0} + \frac{\sigma a}{4\varepsilon_0} = \frac{13\sigma a}{36\varepsilon_0}$$

(d) $\rho_b = -\nabla\cdot\mathbf{P} = 0,$ because $\mathbf{P}$ is position independent.

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \text{slab 1}\begin{cases} \sigma_{\text{up}} = \mathbf{P}_1\cdot\hat{\mathbf{n}}_\uparrow = -\dfrac{8\sigma}{9}\hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = -\dfrac{8\sigma}{9} \\ \sigma_{\text{down}} = \mathbf{P}_1\cdot\hat{\mathbf{n}}_\downarrow = -\dfrac{8\sigma}{9}\hat{\mathbf{z}}\cdot -\hat{\mathbf{z}} = +\dfrac{8\sigma}{9} \end{cases} \\ \text{slab 2}\begin{cases} \sigma_{\text{up}} = \mathbf{P}_2\cdot\hat{\mathbf{n}}_\uparrow = -\dfrac{3\sigma}{4}\hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = -\dfrac{3\sigma}{4} \\ \sigma_{\text{down}} = \mathbf{P}_2\cdot\hat{\mathbf{n}}_\downarrow = -\dfrac{3\sigma}{4}\hat{\mathbf{z}}\cdot -\hat{\mathbf{z}} = +\dfrac{3\sigma}{4} \end{cases} \end{cases}$$

(e) Consider the Gauss's law using the free charge and bound charge.

$\nabla\cdot E = \dfrac{1}{\varepsilon_0}(\rho_{\text{f}} + \rho_b) \;\Rightarrow\; \oint \mathbf{E}\cdot d\mathbf{a} = \dfrac{1}{\varepsilon_0}(Q_{\text{f}} + Q_b)$

In slab 1, $Ea = \dfrac{1}{\varepsilon_0}(\sigma_{\text{f}} + \sigma_{b1\text{u}})a = -\dfrac{\sigma}{9\varepsilon_0}\hat{\mathbf{z}},$

In slab 1, $Ea = \dfrac{1}{\varepsilon_0}(\sigma_{\text{f}} + \sigma_{b1\text{u}} + \sigma_{b1d} + \sigma_{b2u})a = -\dfrac{\sigma}{4\varepsilon_0}\hat{\mathbf{z}},$

The results are the same as (a).

3. Textbook 4.3.3 and 6.3.3

(a)

Normal: $\nabla\cdot\mathbf{D} = \rho_f$. Consider a wafer-thin pillbox. Gauss's law states that $\oint_S \mathbf{D}\cdot d\mathbf{a} = \int_v \rho_f d\tau$.

The sides of the pillbox contribute nothing to the flux, in the limit as the thickness ε goes to zero.

$(D^{\perp}_{above} - D^{\perp}_{below}) = \sigma_f$.

Tangential: $\nabla\times\mathbf{D} = \nabla\times\mathbf{P}$. Consider a thin rectangular loop. The curl of the Ampere's law states that $\oint_P \mathbf{D}\cdot d\ell = \oint_P \mathbf{P}\cdot d\ell$. The ends gives nothing (as ε→0), and the sides give

$(D^{//}_{\text{above}} - D^{//}_{\text{below}})\ell = (P^{//}_{\text{above}} - P^{//}_{\text{below}})\ell \;\Rightarrow\; (D^{//}_{\text{above}} - D^{//}_{\text{below}}) = (P^{//}_{\text{above}} - P^{//}_{\text{below}})$ or

$(\mathbf{D}^{//}_{\text{above}} - \mathbf{D}^{//}_{\text{below}}) = (\mathbf{P}^{//}_{\text{above}} - \mathbf{P}^{//}_{\text{below}})$

(b) **Normal:** $\nabla\cdot\mathbf{H} = -\nabla\cdot\mathbf{M}$. Consider a wafer-thin pillbox. Gauss's law states that

$\oint_S \mathbf{H}\cdot d\mathbf{a} = -\oint_S \mathbf{M}\cdot d\mathbf{a}$. The sides of the pillbox contribute nothing to the flux, in the limit as the thickness ε goes to zero. $H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below})$.

Tangential: $\nabla\times\mathbf{H} = \mathbf{J}_f$. Consider a thin rectangular loop. The curl of the Ampere's law states that $\oint_P \mathbf{H}\cdot d\ell = \mu_0 I_{f\text{enc}}$. The ends gives nothing (as ε→0), and the sides give $(H^{//}_{\text{above}} - H^{//}_{\text{below}})\ell = \mu_0 K_f \ell \quad \Rightarrow \quad H^{//}_{\text{above}} - H^{//}_{\text{below}} = \mu_0 K_f$ or $\mathbf{H}^{//}_{above} - \mathbf{H}^{//}_{below} = \mathbf{K}_f\times\hat{\mathbf{n}}$.

4. Prob. 5.16

(a) According to the boundary conditions, the top plate produces a parallel field $\mu_0 K/2$, pointing out of the page for points above it and into the page for points below) The bottom plate produces a parallel field $\mu_0 K/2$, pointing into the page for points above it and out of the page for points below). Between the plates, the fields add up to $B = \mu_0 K = \mu_0 \sigma v$.

Above and below both plates, the fields cancel $B = 0$.

(b) $d\mathbf{F} = \mathbf{I}d\ell\times\mathbf{B} = \mathbf{K}da\times\mathbf{B} = \mathbf{J}d\tau\times\mathbf{B}$

$d\mathbf{F} = \mathbf{K}da\times\mathbf{B} \;\Rightarrow\; dF = \mathbf{K}\times\mathbf{B}da$

$$\frac{dF}{da} = \mathbf{K}\times\mathbf{B} = \sigma v\frac{\mu_0\sigma v}{2} = \frac{\mu_0\sigma^2 v^2}{2} \text{ (repulsive force per unit area)}$$

(c) The electric force of the plates is attractive $\frac{dF_E}{da} = \sigma\mathbf{E} = \sigma\frac{\sigma}{\varepsilon_0} = \frac{\sigma^2}{\varepsilon_0}$ (attractive force per unit area)

Balance: $\frac{dF}{da} = \frac{d(F_B+F_E)}{da} = \frac{\mu_0\sigma^2 v^2}{2} - \frac{\sigma^2}{2\varepsilon_0} = 0 \quad\Rightarrow\quad v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$ the speed of light.

5.

(a) $\mathbf{M}(s,\phi,z) = M_0\hat{\phi}$ (A/m), $\mathbf{J}_b = \nabla\times\mathbf{M} = [-\frac{\partial M_\phi}{\partial z}]\hat{\mathbf{s}} + \frac{1}{s}[\frac{\partial(sM_\phi)}{\partial s}]\hat{\mathbf{z}} = \frac{M_0}{s}\hat{\mathbf{z}}$

$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = M_0\hat{\phi}\times\hat{\mathbf{n}}$,

$$\begin{cases}\text{outer cylinder } \hat{\mathbf{n}} = \hat{\mathbf{s}} \\ \text{top surface } \hat{\mathbf{n}} = \hat{\mathbf{z}}\end{cases} \Rightarrow \begin{cases}\text{outer cylinder } \mathbf{K}_b = -M_0\hat{\mathbf{z}} \\ \text{top surface } \mathbf{K}_b = M_0\hat{\mathbf{s}}\end{cases}$$

(b) Total magnet field is equal to a complete torus plus a square loop with reverse current.

$\mathbf{B}_{torus} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0\mathbf{M} = \mu_0 M_0\hat{\phi}$

L>>*a*, the contribution from the bound volume current can be omitted $\mathbf{J}_b = \frac{2\pi M_0}{L}\hat{\mathbf{z}} \cong 0$.

$$\mathbf{B}_{loop}(r) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{K}_b(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}da' = \frac{4\mu_0}{4\pi}K_b w\hat{\mathbf{z}}\int_{-a/2}^{a/2}\frac{1}{(\frac{a}{2})^2 + l^2}dl$$

$K_b = -M_0$, let $l = \frac{a}{2}\tan\theta$,

$$\mathbf{B}_{loop}(r) = -\frac{\mu_0}{\pi} M_0 w\hat{\phi}\int_{-\pi/4}^{\pi/4} \frac{\frac{a}{2}\sec\theta}{(\frac{a}{2})^2 \sec^2\theta} d\theta = -\frac{2\mu_0}{a\pi} M_0 w\hat{\phi}\cdot\sin\theta\Big|_{-\pi/4}^{\pi/4} = -\frac{2\sqrt{2}\mu_0}{a\pi} M_0 w\hat{\phi}$$

$$\mathbf{B} = \mathbf{B}_{torus} + \mathbf{B}_{loop} = \mu_0 M_0\hat{\phi} - \frac{2\sqrt{2}\mu_0}{a\pi} M_0 w\hat{\phi} = \mu_0 M_0 (1 - \frac{2\sqrt{2}w}{a\pi})\hat{\phi}$$